Quadrocopter Ball Juggling

Appendix: Trajectory Generation

Mark Müller, Sergei Lupashin and Raffaello D'Andrea
July 5, 2011

The quadrocopter's position in space, under the assumption that the angles remain small, can be written as:

$$\ddot{\mathbf{s}}_r = \mathbf{n}f + \mathbf{g} = (\theta,\, -\phi,\, 1) \cdot f + (0,\, 0,\, -g)\,. \tag{1}$$

For convenience, we drop the distinction between desired and actual values. Substituting the affine inputs

$$f(t) = A_f t + B_f \tag{2}$$

$$\ddot{\theta}(t) = A_\theta t + B_\theta \tag{3}$$

$$\ddot{\phi}(t) = A_\phi t + B_\phi, \tag{4}$$

into (1), we can write the racket normal's second time derivative as a polynomial in time, which we integrate twice. Then the quadrocopter's accelerations along each axis are also simply polynomials in time, which we integrate to get velocity and position.

The time is taken as $t = 0$ at the start of the trajectory, and the impact time occurs at $t = T$. The following conditions must be met at impact, where $\mathbf{s}_f$ is the impact position, $V_\perp$ is the desired racket speed in the direction of the normal, and the desired racket orientation is given by $\mathbf{n}_f$:

$$\mathbf{s}_r(T) = \mathbf{s}_f \tag{5}$$

$$\mathbf{n}(T) = \mathbf{n}_f \tag{6}$$

$$\dot{\mathbf{s}}_r(T)^T \mathbf{n}(T) = V_\perp. \tag{7}$$

Introducing the following notation:

$$(x_1(t),\, x_2(t),\, x_3(t)) := \mathbf{s}_r(t) \tag{8}$$

$$(n_{f_1},\, n_{f_2},\, n_{f_3}) := \mathbf{n}_f \tag{9}$$

$$x_{0_i} := x_i(0),\ i \in \{1, 2, 3\} \tag{10}$$

$$\dot{x}_{0_i} := \dot{x}_i(0),\ i \in \{1, 2, 3\} \tag{11}$$

$$x_{f_i} := x_i(T),\ i \in \{1, 2, 3\} \tag{12}$$

$$\theta_0 := \theta(0) \tag{13}$$

$$\dot{\theta}_0 := \dot{\theta}(0) \tag{14}$$

$$\theta_f := \theta(T) \tag{15}$$

$$\phi_0 := \phi(0) \tag{16}$$

$$\dot{\phi}_0 := \dot{\phi}(0) \tag{17}$$

$$\phi_f := \phi(T) \tag{18}$$

$$\Delta x_1 := x_{f_1} - x_{0_1} - \dot{x}_{0_1} T \tag{19}$$

$$\Delta x_2 := x_{f_2} - x_{0_2} - \dot{x}_{0_2} T \tag{20}$$

$$\Delta x_3 := x_{f_3} - x_{0_3} - \dot{x}_{0_3} T + \frac{1}{2} g T^2 \tag{21}$$

$$\Delta\theta := \theta_f - \theta_0 - \dot{\theta}_0 T \tag{22}$$

$$\Delta\phi := \phi_f - \phi_0 - \dot{\phi}_0 T. \tag{23}$$

We now rewrite (5) - (7) as the following six scalar equations:

$$x_i(T) = x_{f_i}, \;\; i = 1, 2, 3 \tag{24}$$

$$\theta_f = \arctan\left(\frac{n_{f_1}}{n_{f_3}}\right) \tag{25}$$

$$\phi_f = \arcsin\left(n_{f_2}\right) \tag{26}$$

$$\dot{\mathbf{s}}_r(T)^T \mathbf{n}_f = \Delta V. \tag{27}$$

Substituting the polynomials resulting from the integration of (1), we get the following six equations:

$$\begin{aligned}\Delta x_1 - \theta_0 \Delta x_3 - \tfrac{1}{3}\dot{\theta}_0 \Delta x_3 T - \tfrac{1}{6}\Delta\theta\Delta x_3 = \left(\tfrac{1}{36}\dot{\theta}_0 T^4 + \tfrac{1}{45}\Delta\theta\right) A_f ... \\ + \left(-\tfrac{1}{90}\Delta x_3 T^3\right) A_\theta + \left(-\tfrac{1}{1080} T^6\right) A_f A_\theta\end{aligned} \tag{28}$$

$$\begin{aligned}-\Delta x_2 - \phi_0 \Delta x_3 - \tfrac{1}{3}\dot{\phi}_0 \Delta x_3 T - \tfrac{1}{6}\Delta\phi\Delta x_3 = \left(\tfrac{1}{36}\dot{\phi}_0 T^4 + \tfrac{1}{45}\Delta\phi\right) A_f ... \\ + \left(-\tfrac{1}{90}\Delta x_3 T^3\right) A_\phi + \left(-\tfrac{1}{1080} T^6\right) A_f A_\phi\end{aligned} \tag{29}$$

$$\begin{aligned}-\left(\dot{x}_{0_1} T - 2\theta_0 \Delta x_3 - \tfrac{1}{3}\dot{\theta}_0 \Delta x_3 T + 4\Delta x_1\right) n_{f_1} - \left(\dot{x}_{0_2} T + 2\phi_0 \Delta x_3 + \tfrac{1}{3}\dot{\phi}_0 \Delta x_3 T + 4\Delta x_2\right) n_{f_2} ... \\ - \left(\dot{x}_{0_3} T - gT^2 - 2\Delta x_3\right) n_{f_3} + \Delta V T = \left(\tfrac{1}{60}\Delta x_3 T^3 n_{f_1}\right) A_\theta + \left(-\tfrac{1}{60}\Delta x_3 T^3 n_{f_2}\right) A_\phi ... \\ + \left(\left(\tfrac{1}{6}\theta_0 T^3 + \tfrac{1}{18}\dot{\theta}_0 T^4 + \tfrac{1}{20}\Delta\theta T^3\right) n_{f_1} + \left(-\tfrac{1}{6}\phi_0 T^3 - \tfrac{1}{18}\dot{\phi}_0 T^4 - \tfrac{1}{20}\Delta\phi T^3\right) n_{f_2} + \left(\tfrac{1}{6}T^3\right) n_{f_3}\right) A_f ...\end{aligned} \tag{30}$$

$$B_f = \frac{1}{T^2}\left(2\Delta x_3 - \frac{1}{3}A_f T^3\right) \tag{31}$$

$$B_\theta = \frac{1}{T^2}\left(2\Delta\theta - \frac{1}{3}A_\theta T^3\right) \tag{32}$$

$$B_\phi = \frac{1}{T^2}\left(2\Delta\phi - \frac{1}{3}A_\phi T^3\right) \tag{33}$$

By (31)-(33) we can see that B_f, B_θ and B_ϕ are affine functions in A_f, A_θ and A_ϕ, respectively. This means that if we select a set of solutions based on "small" values for A_f, A_θ and A_ϕ, it will also imply small values for B_f, B_θ and B_ϕ, respectively.

For brevity we introduce the coefficients b_i, c_i, d_i, f_i and g_i. By substitution, we can rewrite (28) - (33) as three equations in three unknowns:

$$0 = b_0 + b_1 A_f + b_2 A_\theta + b_3 A_f A_\theta \tag{34}$$

$$0 = c_0 + c_1 A_f + c_2 A_\phi + c_3 A_f A_\phi \tag{35}$$

$$0 = d_0 + d_1 A_f + d_2 A_\theta + d_3 A_\phi \tag{36}$$

where

$$b_0 := \Delta x_1 - \theta_0 \Delta x_3 - \frac{1}{3}\dot{\theta}_0 \Delta x_3 T - \frac{1}{6}\Delta x_3 \Delta\theta \tag{37}$$

$$b_1 := \left(\frac{1}{36}\dot{\theta}_0 T + \frac{1}{45}\Delta\theta\right) T^3 \tag{38}$$

$$b_2 := \left(-\frac{1}{90}\Delta x_3\right) T^3 \tag{39}$$

$$b_3 := -\frac{1}{1080} T^6 \tag{40}$$

$$c_0 := -\Delta x_2 - \phi_0 \Delta x_3 - \frac{1}{3}\dot{\phi}_0 \Delta x_3 T - \frac{1}{6}\Delta x_3 \Delta\phi \tag{41}$$

$$c_1 := \left(\frac{1}{36}\dot{\phi}_0 T + \frac{1}{45}\Delta\phi\right) T^3 \tag{42}$$

$$c_2 := \left(-\frac{1}{90}\Delta x_3\right) T^3 \tag{43}$$

$$c_3 := -\frac{1}{1080} T^6 \tag{44}$$

$$\begin{aligned} d_0 := V_\perp T &- \left(\dot{x}_{0_1} T - 2\Delta x_3 \theta_0 - \frac{1}{3}\dot{\theta}_0 \Delta x_3 T + 4\Delta x_1\right) n_{f_1} \\ &- \left(\dot{x}_{0_2} T + 2\Delta x_3 \phi_0 + \frac{1}{3}\dot{\phi}_0 \Delta x_3 T + 4\Delta x_2\right) n_{f_2} \\ &- \left(\dot{x}_{0_3} T + 2\Delta x_3 - gT^2\right) n_{f_3} \end{aligned} \tag{45}$$

$$\begin{aligned} d_1 := &\left(\frac{1}{6}\theta_0 T^3 + \frac{1}{18}\dot{\theta}_0 T^4 + \frac{1}{20}\Delta\theta T^3\right) n_{f_1} \\ &+ \left(-\frac{1}{6}\phi_0 T^3 - \frac{1}{18}\dot{\phi}_0 T^4 - \frac{1}{20}\Delta\phi T^3\right) n_{f_2} \\ &+ \left(\frac{1}{6} T^3\right) n_{f_3} \end{aligned} \tag{46}$$

$$d_2 := \left(\frac{1}{60}\Delta x_3 T^3\right) n_{f_1} \tag{47}$$

$$d_3 := \left(-\frac{1}{60}\Delta x_3 T^3\right) n_{f_2} \tag{48}$$

Simplifying, we get two equations in two unknowns:

$$0 = e_0 + e_1 A_\theta + e_2 A_\phi + e_3 A_\theta A_\phi + e_4 A_\theta^2 \tag{49}$$

$$0 = f_0 + f_1 A_\phi + f_2 A_\theta + f_3 A_\theta A_\phi + f_4 A_\phi^2 \tag{50}$$

where

$$e_0 = b_0 d_1 - b_1 d_0 \tag{51}$$

$$e_1 = -b_1 d_2 + b_2 d_1 + b_3 d_0 \tag{52}$$

$$e_2 = -b_1 d_3 \tag{53}$$

$$e_3 = -b_3 d_3 \tag{54}$$

$$e_4 = -b_3 d_2 \tag{55}$$

$$f_0 = c_0 d_1 - c_1 d_0 \tag{56}$$

$$f_1 = -c_1 d_3 + c_2 d_1 + c_3 d_0 \tag{57}$$

$$f_2 = -c_1 d_2 \tag{58}$$

$$f_3 = -c_3 d_1 \tag{59}$$

$$f_4 = -c_3 d_3 \tag{60}$$

which can be further reduced to one equation in one unknown:

$$0 = g_0 + g_1 A\theta^4 + g_2 A_\theta^3 + g_3 A_\theta^2 + g_4 A_\theta \tag{61}$$

with

$$g_1 = e_4 e_4 f_4 - e_3 e_4 f_3 \tag{62}$$

$$g_2 = e_3 e_3 f_2 - e_1 e_3 f_3 - e_3 e_4 f_1 + 2 e_1 e_4 f_4 - e_2 e_4 f_3 \tag{63}$$

$$g_3 = e_1 e_1 f_4 - e_3 e_3 f_0 - e_1 e_3 f_1 + e_0 e_3 f_3 - e_1 e_2 f_3 + 2 e_2 e_3 f_2 - e_2 e_4 f_1 - 2 e_0 e_4 f_4 \tag{64}$$

$$g_4 = e_2 e_2 f_2 + e_0 e_3 f_1 - e_1 e_2 f_1 - 2 e_0 e_1 f_4 + e_0 e_2 f_3 - 2 e_2 e_3 f_0 \tag{65}$$

$$g_0 = f_4 e_0 e_0 + f_1 e_0 e_2 - f_0 e_2 e_2 \tag{66}$$

This quartic equation can be solved analytically. For an example of a C++ implementation, see http://marcusbannerman.co.uk/index.php/component/content/article/42-articles/87-quartic-and-cubic-root-finder-in-c.html.