Cooperative Quadrocopter Ball Throwing and Catching Online Appendix

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I. Minimum Maximum Acceleration Trajectory

A. Maneuver Structure

In order to find a maneuver that minimizes the maximum acceleration of a one-dimensional system with constrained jerk, we have to solve the optimizing problem

$$\begin{aligned} \text{minimize} \quad & \max(|a|) \\ \text{subject to} \quad & \dot{\mathbf{s}} = f(\mathbf{s}, k), \\ & \mathbf{s}(t_0) = \mathbf{s}_0, \\ & \mathbf{s}(t_f) = \mathbf{s}_f, \\ & k \in [-k_{max}, k_{max}] \; \forall t \in [t_0, t_f], \end{aligned} \tag{1}$$

with the system dynamics

$$\dot{\mathbf{s}} = f(\mathbf{s}, k) = (\dot{p}, \dot{v}, \dot{a}) = (v, a, k). \tag{2}$$

We use Pontryagin's minimum principle to determine the structure of the optimal solution. The maximum acceleration magnitude within the interval $[t_0, t_f]$ can be written as

$$\max(|a|) = \lim_{n \to \infty} \sqrt[2n]{\frac{1}{t_f - t_0} \int_{t_0}^{t_f} a^{2n} dt}, \tag{3}$$

where n is a positive integer. Since the function $\sqrt[2n]{1/(t_f - t_0)(.)}$ is strictly increasing, we can minimize its argument instead of minimizing the function value. Therefore, we define the cost function to be

$$J = \int_{t_0}^{t_f} g(\mathbf{s}, k) dt = \int_{t_0}^{t_f} a^{2n} dt \quad \Rightarrow \quad g(\mathbf{s}, k) = a^{2n}, \tag{4}$$

where we later increase n towards infinity. The Hamiltonian is then given by

$$H(\mathbf{s}, k, \mathbf{p}) = g(\mathbf{s}, k) + \mathbf{p}^T f(\mathbf{s}, k) = a^{2n} + p_1 v + p_2 a + p_3 k, \tag{5}$$

where $\mathbf{p}$ denotes the costates. We can see that the Hamiltonian is linear in k, meaning that the optimal control input jumps between $\pm k_{max}$ if p_3 switches its sign (the optimal solution minimizes the Hamiltonian). Additionally, for each switch a singular arc might occur, meaning that p_3 stays at zero for a nontrivial interval of time. Within these intervals, the input k is determined by the condition that p_3 and all its time-derivatives stay at zero. The adjoint equation

$$\dot{\mathbf{p}} = -\nabla_{\mathbf{s}} H(\mathbf{s}, k, \mathbf{p}) \tag{6}$$

yields for our problem:

$$\begin{aligned} \dot{p}_1 &= 0 && \Rightarrow \quad p_1 = c_1 \\ \dot{p}_2 &= -p_1 && \Rightarrow \quad p_2 = c_2 - c_1 t \\ \dot{p}_3 &= -p_2 - 2na^{2n-1} && \end{aligned} \tag{7}$$

By solving $\dot{p}_3 = 0$ for a, we get the acceleration within a singular arc:

$$a(t) = \pm c_1 \left(\pm \frac{t - c_2/c_1}{2n} \right)^{\frac{1}{2n-1}}. \tag{8}$$

For $n > 1$, the imaginary part of (8) only vanishes for positive values within the brackets, we therefore factor out $\pm c_1$ such that the bracket is positive. To find the acceleration within a singular interval of a maneuver that minimizes $\max(|a|)$, we increase n towards infinity, yielding

$$\lim_{n\to\infty} \pm c_1 \left(\pm \frac{t - c_2/c_1}{2n} \right)^{\frac{1}{2n-1}} = \pm c_1. \tag{9}$$

Hence, the acceleration is constant if we are in a singular arc, and consequently the jerk stays at zero. Furthermore, knowing that the acceleration is constant, it can be verified that more than two switches always increase the required maximum acceleration.

Summing up, the optimal jerk trajectory switches twice between the jerk boundaries, and at each switch it might stay at zero for a nontrivial interval of time; the maneuver has at most five intervals:

- $[t_0, t_1)$: $k = \pm k_{max}$,
- $[t_1, t_2)$: $k = 0$ and $a = \pm a_{max}$,
- $[t_2, t_3)$: $k = \mp k_{max}$,
- $[t_3, t_4)$: $k = 0$ and $a = \mp a_{max}$,
- $[t_4, t_f]$: $k = \pm k_{max}$,

with $a_{max} = c_1$.

B. Maneuver Parameters

Each of the two singular intervals might vanish, hence we do not know the number of intervals beforehand. Furthermore, the sign of the jerk within the first interval is not known, either. We therefore compute the solutions for all possible cases, and pick the right solution at the end. The right solution is the one with the smallest a_{max} among all solutions that make physically sense.

1) Case $t_2 > t_1$ and $t_4 > t_3$: In order to find the parameters of such a maneuver, we have to find the five unknowns

$$t_1, t_2, t_3, t_4, a_{max}, \tag{10}$$

satisfying the five conditions

$$p(t_f) = p_f, \tag{11}$$

$$v(t_f) = v_f, \tag{12}$$

$$a(t_f) = a_f, \tag{13}$$

$$a(t_1) = a_{max}, \tag{14}$$

$$a(t_3) = -a_{max}. \tag{15}$$

By inspecting the structure of the acceleration trajectory, the conditions (13)-(15) can be used to express the unknowns t_1, t_3, and t_4 as linear functions of the two unknowns a_{max} and t_2:

$$\begin{aligned} t_1 &= t_0 + \tfrac{a_{max} - a_0}{k_{max}}, \\ t_3 &= t_2 + 2\tfrac{a_{max}}{k_{max}}, \\ t_4 &= t_f - \tfrac{a_{max} + a_f}{k_{max}}. \end{aligned} \tag{16}$$

In order to obtain the closed-form solutions of the maneuver parameters, we integrate the different maneuver intervals analytically, and subsequently substitute the unknowns t_1, t_3, and t_4 by inserting (16). We get

$$
\begin{aligned}
p(t_f) = & \frac{-a_0{}^3-24a_{max}{}^3+a_f{}^3-3a_0{}^2(a_{max}+k_{max}T)+3a_{max}{}^2(a_f+7k_{max}T-8k_{max}T_2)+3a_0a_{max}(7a_{max}-2k_{max}T+4k_{max}T_2)}{6k_{max}{}^2} \\
& +\frac{3a_{max}\left(a_f{}^2-k_{max}{}^2\left(T^2-4TT_2+2T_2{}^2\right)\right)+6k_{max}{}^2(p_0+Tv_0)}{6k_{max}{}^2}, \qquad (17)
\end{aligned}
$$

$$
v(t_f) = \frac{-a_0{}^2-2a_0a_{max}+8a_{max}{}^2+a_f{}^2+2a_{max}(a_f-k_{max}T+2k_{max}T_2)+2k_{max}v_0}{2k_{max}}, \qquad (18)
$$

$$
a(t_f) = a_f, \qquad (19)
$$

where the notations $T_2 = t_2 - t_1$ and $T = t_f - t_0$ were used. The resulting final velocity (18) is linear in T_2, hence we can insert (18) into condition (12) and subsequently solve for T_2:

$$
T_2 = \frac{a_0{}^2+2a_0a_{max}-8a_{max}{}^2-2a_{max}a_f-a_f{}^2+2a_{max}k_{max}T-2k_{max}v_0+2k_{max}v_f}{4a_{max}k_{max}}. \qquad (20)
$$

By inserting (20) into (17), the final position condition (11) results in a cubic equation for the remaining unknown a_{max}:

$$
c_3a_{max}{}^3 + c_2a_{max}{}^2 + c_1a_{max} + c_0 = 0, \qquad (21)
$$

with

$$
\begin{aligned}
c_3 &= -\frac{a_0-a_f+k_{max}T}{2k_{max}{}^2}, \\
c_2 &= \frac{(a_0-a_f+k_{max}T)^2}{4k_{max}{}^2}, \\
c_1 &= \frac{a_0{}^3-a_f{}^3-3a_f{}^2k_{max}T+3a_0{}^2(a_f-k_{max}T)-3a_0\left(a_f{}^2+2k_{max}(v_0-v_f)\right)+6a_fk_{max}(v_f-v_0)+6k_{max}{}^2(2p_0-2p_f+T(v_0+v_f))}{12k_{max}{}^2}, \\
c_0 &= -\frac{\left(a_0{}^2-a_f{}^2+2k_{max}(v_f-v_0)\right)^2}{16k_{max}{}^2}.
\end{aligned} \qquad (22)
$$

This equation can be solved using Cardano's method for general cubic equations, yielding three possible solutions for a_{max}. Afterwards, we can use (16) and (20) to obtain the other four unknowns.

2) Case $t_2 = t_1$ and $t_4 > t_3$: In order to find the maneuver parameters for this case, we have to find the four unknowns

$$
t_1, t_3, t_4, a_{max}, \qquad (23)
$$

satisfying the four conditions

$$
p(t_f) = p_f, \qquad (24)
$$

$$
v(t_f) = v_f, \qquad (25)
$$

$$
a(t_f) = a_f, \qquad (26)
$$

$$
a(t_3) = -a_{max}. \qquad (27)
$$

Similar to the derivations above, we can use the conditions (26) and (27) to express t_1 and t_4 as linear functions of t_3 and a_{max}:

$$
\begin{aligned}
t_1 &= \tfrac{1}{2}t_0 + \tfrac{1}{2}t_3 - \frac{a_{max}+a_0}{2k_{max}}, \\
t_4 &= t_f - \frac{a_{max}+a_f}{k_{max}}.
\end{aligned} \qquad (28)
$$

Again, we integrate the system over all intervals and insert the substitution (28). We get for the final state:

$$
\begin{aligned}
p(t_f) =& \frac{-a_0{}^3+a_f{}^3+6k_{max}{}^2p_0+3a_{max}{}^2(a_f-k_{max}T)-3a_0{}^2(a_{max}+k_{max}T)+6k_{max}{}^3TT_3{}^2-6k_{max}{}^3T_3{}^3}{6k_{max}{}^2} \\
&+\frac{3a_{max}\left(a_f{}^2-k_{max}{}^2\left(T^2-2T_3{}^2\right)\right)-3a_0\left(a_{max}{}^2+2a_{max}k_{max}T-2k_{max}{}^2T_3{}^2\right)+6k_{max}{}^2Tv_0}{6k_{max}{}^2}, \qquad (29)\\
v(t_f) =& \frac{-a_0{}^2-2a_0a_{max}+a_f{}^2+2a_{max}(a_f-k_{max}T)+2k_{max}{}^2T_3{}^2+2k_{max}v_0}{2k_{max}}, \qquad (30)\\
a(t_f) =& a_f, \qquad (31)
\end{aligned}
$$

with $T_3 = t_3 - t_2$. In this case, the final velocity (30) is quadratic with respect to T_3, and we get two solutions when we solve condition (25):

$$
T_3 = \pm\frac{\sqrt{a_0{}^2+2a_0a_{max}-2a_{max}a_f-a_f{}^2+2a_{max}k_{max}T-2k_{max}v_0+2k_{max}v_f}}{\sqrt{2}k_{max}}. \qquad (32)
$$

Since T_3 must be positive for a valid trajectory, we insert the positive solution of (32) into (29), and the final position condition (24) yields a radical equation:

$$
c_2a_{max}{}^2 + c_1a_{max} + c_0 + \sqrt{d_3a_{max}{}^3 + d_2a_{max}{}^2 + d_1a_{max} + d_0} = 0, \qquad (33)
$$

with

$$
\begin{aligned}
c_2 &= \frac{a_0-a_f+k_{max}T}{2k_{max}{}^2},\\
c_1 &= \frac{2a_0{}^2-2a_0(a_f-k_{max}T)+k_{max}\left(-2a_fT+k_{max}T^2-2v_0+2v_f\right)}{2k_{max}{}^2},\\
c_0 &= \frac{2a_0{}^3+a_f{}^3-3a_f{}^2k_{max}T-3a_0\left(a_f{}^2+2k_{max}(v_0-v_f)\right)+6k_{max}{}^2(p_0-p_f+Tv_f)}{6k_{max}{}^2},\\
d_3 &= \frac{(a_0-a_f+k_{max}T)^3}{k_{max}{}^4},\\
d_2 &= \frac{3(a_0-a_f+k_{max}T)^2\left(a_0{}^2-a_f{}^2+2k_{max}(-v_0+v_f)\right)}{2k_{max}{}^4},\\
d_1 &= \frac{3(a_0-a_f+k_{max}T)\left(a_0{}^2-a_f{}^2+2k_{max}(-v_0+v_f)\right)^2}{4k_{max}{}^4},\\
d_0 &= \frac{\left(a_0{}^2-a_f{}^2+2k_{max}(-v_0+v_f)\right)^3}{8k_{max}{}^4}.
\end{aligned} \qquad (34)
$$

Equation (33) can be converted to a quartic equation

$$
c_2^2a_{max}{}^4 + (2c_1c_2 + d_3)a_{max}{}^3 + (c_1^2 + 2c_0c_2 + d_2)a_{max}{}^2 + (2c_0c_1 + d_1)a_{max} + c_0^2 + d_0 = 0, \qquad (35)
$$

which can be solved for a general case using Ferrari's method. Four possible solutions for a_{max} exist.

3) Case $t_2 > t_1$ and $t_4 = t_3$: In order to find the parameters of such a maneuver, we have to find the four unknowns

$$
t_1, t_2, t_3, a_{max}, \qquad (36)
$$

satisfying the four conditions

$$
p(t_f) = p_f, \qquad (37)
$$

$$
v(t_f) = v_f, \qquad (38)
$$

$$
a(t_f) = a_f, \qquad (39)
$$

$$
a(t_1) = a_{max}. \qquad (40)
$$

Again, we can use the conditions (39) and (40) to express t_1 and t_2 as linear functions of t_3 and a_{max}:

$$
\begin{aligned}
t_1 &= t_0 + \tfrac{a_{max}-a_0}{k_{max}},\\
t_2 &= 2t_3 - t_f + \tfrac{a_f-a_{max}}{k_{max}}.
\end{aligned} \tag{41}
$$

Integrating the system over all intervals, and subsequently applying substitution (41) yields

$$
\begin{aligned}
p(t_f) =& \tfrac{-a_0{}^3+a_f{}^3+6k_{max}{}^2p_0-3a_0a_{max}(a_{max}-2k_{max}T)+3a_0{}^2(a_{max}-k_{max}T)+3a_{max}{}^2(a_f-k_{max}T)}{6k_{max}{}^2}\\
&+\tfrac{a_{max}\left(-3a_f{}^2+3k_{max}{}^2\left(T^2+2T_3{}^2\right)\right)+6k_{max}{}^2Tv_0-6a_fk_{max}{}^2T_3{}^2-6k_{max}{}^3T_3{}^3}{6k_{max}{}^2},
\end{aligned} \tag{42}
$$

$$
v(t_f) = \tfrac{-a_0{}^2+2a_0a_{max}-2a_{max}a_f+a_f{}^2+2a_{max}k_{max}T-2k_{max}{}^2T_3{}^2+2k_{max}v_0}{2k_{max}}, \tag{43}
$$

$$
a(t_f) = a_f. \tag{44}
$$

Again, the final velocity (43) is quadratic in T_3, and solving the final velocity condition (38) for T_3 yields:

$$
T_3 = \pm\tfrac{\sqrt{-a_0{}^2+2a_0a_{max}-2a_{max}a_f+a_f{}^2+2a_{max}k_{max}T+2k_{max}v_0-2k_{max}v_f}}{\sqrt{2}k_{max}}. \tag{45}
$$

We insert the positive solution of (45) into (42), and the final position condition (37) yields a radical equation:

$$
c_2a_{max}{}^2 + c_1a_{max} + c_0 + \sqrt{d_3a_{max}{}^3 + d_2a_{max}{}^2 + d_1a_{max} + d_0} = 0, \tag{46}
$$

with

$$
\begin{aligned}
c_2 &= \tfrac{a_0-a_f+k_{max}T}{2k_{max}{}^2},\\
c_1 &= \tfrac{2a_f{}^2-2a_fk_{max}T-2a_0(a_f-k_{max}T)+k_{max}\left(k_{max}T^2+2v_0-2v_f\right)}{2k_{max}{}^2},\\
c_0 &= \tfrac{-a_0{}^3-2a_f{}^3+3a_0{}^2(a_f-k_{max}T)+6k_{max}{}^2(p_0-p_f+Tv_0)+6a_fk_{max}(-v_0+v_f)}{6k_{max}{}^2},\\
d_3 &= \tfrac{(a_0-a_f+k_{max}T)^3}{k_{max}{}^4},\\
d_2 &= \tfrac{3(a_0-a_f+k_{max}T)^2\left(-a_0{}^2+a_f{}^2+2k_{max}(v_0-v_f)\right)}{2k_{max}{}^4},\\
d_1 &= \tfrac{3(a_0-a_f+k_{max}T)\left(a_0{}^2-a_f{}^2+2k_{max}(-v_0+v_f)\right)^2}{4k_{max}{}^4},\\
d_0 &= \tfrac{\left(-a_0{}^2+a_f{}^2+2k_{max}(v_0-v_f)\right)^3}{8k_{max}{}^4}.
\end{aligned} \tag{47}
$$

Finally, we convert (46) to a quartic equation

$$
c_2^2a_{max}{}^4 + (2c_1c_2+d_3)a_{max}{}^3 + (c_1^2+2c_0c_2+d_2)a_{max}{}^2 + (2c_0c_1+d_1)a_{max} + c_0^2 + d_0 = 0, \tag{48}
$$

which can be solved for general cases. Again, four possible solutions for a_{max} result.

4) Case $t_2 = t_1$ and $t_4 = t_3$: In this case, we have two unknowns (t_1 and t_3) and three final state conditions; we cannot find a solution in general, meaning that this case only occurs as limit case of one of the other solutions above.

5) Negative Initial Jerk: All derivations above were done for positive jerk in the first interval. For negative initial jerk, we substitute $k_{max} = -k_{max}$ and $a_{max} = -a_{max}$ and apply the same formulas.

II. Throw Trajectory

A. First Interval, $t_0 < t \leq t_1$

Within the first interval of the throw, the acceleration is constant and given by

$$\begin{aligned} \ddot{r}_{net}(t) &= a_{th1} \sin(\phi_0), \\ \ddot{z}_{net}(t) &= a_{th1} \cos(\phi_0) - g, \end{aligned} \tag{49}$$

where $\phi_0 = \phi(t_0)$ is the steady-state roll angle. The velocity and position trajectories yield:

$$\begin{aligned} \dot{r}_{net}(t) &= a_{th1} \sin(\phi_0) t, \\ r_{net}(t) &= r_0 + \tfrac{1}{2} a_{th1} \sin(\phi_0) t^2, \\ \dot{z}_{net}(t) &= (a_{th1} \cos(\phi_0) - g) t, \\ z_{net}(t) &= z_0 + \tfrac{1}{2}(a_{th1} \cos(\phi_0) - g) t^2, \end{aligned} \tag{50}$$

where $r_0 = r_{net}(t_0)$ denotes the initial net radius, and $z_0 = z_{net}(t_0)$ the initial net reference height.

B. Second Interval, $t_1 < t \leq t_2$

Within the second interval, the vehicles turn outwards and the acceleration trajectories are

$$\begin{aligned} \ddot{r}_{net}(t) &= a_{th2} \sin(\phi_0 + \dot{\phi}_{th2}(t - t_1)), \\ \ddot{z}_{net}(t) &= a_{th2} \cos(\phi_0 + \dot{\phi}_{th2}(t - t_1)) - g. \end{aligned} \tag{51}$$

By integration of the acceleration trajectories, we get

$$\begin{aligned} \dot{r}_{net}(t) &= \dot{r}_{net}(t_1) + \tfrac{a_{th2}(\cos(\phi_0) - \cos(\phi_0 + \dot{\phi}_{th2}(t-t_1)))}{\dot{\phi}_{th2}}, \\ r_{net}(t) &= r_{net}(t_1) + \dot{r}_{net}(t_1)(t - t_1) + \tfrac{a_{th2}(\sin(\phi_0) - \sin(\phi_0 + \dot{\phi}_{th2}(t-t_1)) + \dot{\phi}_{th2}(t-t_1)\cos(\phi_0))}{\dot{\phi}_{th2}^2}, \\ \dot{z}_{net}(t) &= \dot{z}_{net}(t_1) - g(t - t_1) - \tfrac{a_{th2}(\sin(\phi_0) - \sin(\phi_0 + \dot{\phi}_{th2}(t-t_1)))}{\dot{\phi}_{th2}}, \\ z_{net}(t) &= z_{net}(t_1) + \dot{z}_{net}(t_1)(t - t_1) - \tfrac{1}{2} g (t - t_1)^2 + \tfrac{a_{th2}(\cos(\phi_0) - \cos(\phi_0 + \dot{\phi}_{th2}(t-t_1)) - \dot{\phi}_{th2}(t-t_1)\sin(\phi_0))}{\dot{\phi}_{th2}^2}. \end{aligned} \tag{52}$$

The parameters $\dot{\phi}_{th2}$ and t_2 are determined by the two conditions

$$\begin{aligned} r_{net}(t_2) &= l_{net}, \\ \phi(t_2) &= \pi/2. \end{aligned} \tag{53}$$

We can insert (52) into (53) and solve for the unknowns. Two solutions exist; we pick the one yielding a positive roll rate:

$$\begin{aligned} \dot{\phi}_{th2} &= \tfrac{-2\dot{r}_{net}(t_1)\phi_0 + \dot{r}_{net}(t_1)\pi + \sqrt{(\dot{r}_{net}(t_1)\pi - 2\dot{r}_{net}(t_1)\phi_0)^2 + 8 a_{th2}(l_{net} - r_{net}(t_1))((\pi - 2\phi_0)\cos(\phi_0) + 2\sin(\phi_0) - 2)}}{4(l_{net} - r_{net}(t_1))}, \\ t_2 &= \tfrac{\pi/2 - \phi_0}{\dot{\phi}_{th2}}. \end{aligned} \tag{54}$$

C. Third Interval, $t_2 < t \leq t_3$

The trajectories during the stretching interval follow the trajectories of a linear spring-mass system, hence the accelerations are given by

$$\begin{aligned} \ddot{r}_{net}(t) &= -a_{stretch} \sin\left(\lambda_n(t-t_2)+\tau_{stretch}\right), \\ \ddot{z}_{net}(t) &= -g, \end{aligned} \tag{55}$$

where λ_n is the natural frequency of the system. The velocity and position trajectories yield

$$\begin{aligned} \dot{r}_{net}(t) &= \tfrac{a_{stretch}}{\lambda_n} \cos\left(\lambda_n(t-t_2)+\tau_{stretch}\right), \\ r_{net}(t) &= \tfrac{a_{th3}}{\lambda_n{}^2} + \tfrac{a_{stretch}}{\lambda_n{}^2} \sin\left(\lambda_n(t-t_2)+\tau_{stretch}\right), \\ \dot{z}_{net}(t) &= \dot{z}_{net}(t_2) - g(t-t_2), \\ z_{net}(t) &= z_{net}(t_2) + \dot{z}_{net}(t_2)(t-t_2) - \tfrac{1}{2}g(t-t_2)^2. \end{aligned} \tag{56}$$

We can insert (56) into the three conditions

$$\begin{aligned} r_{net}(t_2{}^+) &= r_{net}(t_2{}^-) = l_{net}, \\ \dot{r}_{net}(t_2{}^+) &= \dot{r}_{net}(t_2{}^-), \\ r_{net}(t_3{}^-) &= r_{net}(t_3{}^+) = l_{net}, \end{aligned} \tag{57}$$

and solve for the three unknowns $a_{stretch}$, $\tau_{stretch}$, and t_3. Since multiple solutions exist, we pick the one with $t_3 > t_2$ and $\tau_{stretch} > 0$:

$$\begin{aligned} a_{stretch} &= -\sqrt{\dot{r}_{net}(t_2)^2\lambda_n{}^2 + \left(a_{th3} - \lambda_n{}^2 l_{net}\right)^2}, \\ \tau_{stretch} &= \arccos\left(-\tfrac{\dot{r}_{net}(t_2)\lambda_n}{\sqrt{\dot{r}_{net}(t_2)^2\lambda_n{}^2 + (a_{th3}-\lambda_n{}^2 l_{net})^2}}\right), \\ t_3 &= t_2 + \tfrac{1}{\lambda_n}\left(\tau_{stretch} - \arcsin\left(\tfrac{a_{th3}-\lambda_n{}^2 l_{net}}{\sqrt{\dot{r}_{net}(t_2)^2\lambda_n{}^2 + (a_{th3}-\lambda_n{}^2 l_{net})^2}}\right)\right). \end{aligned} \tag{58}$$

D. Fourth Interval, $t_3 < t \leq t_f$

In the fourth interval, the vehicles turn back and decelerate. The acceleration trajectories are given by

$$\begin{aligned} \ddot{r}_{net}(t) &= a_{th4} \sin\left(\pi/2 + \dot{\phi}_{th2}(t-t_3)\right), \\ \ddot{z}_{net}(t) &= a_{th4} \cos\left(\pi/2 + \dot{\phi}_{th2}(t-t_3)\right) - g. \end{aligned} \tag{59}$$

The position and velocity trajectories yield

$$\begin{aligned} \dot{r}_{net}(t) &= \dot{r}_{net}(t_3) + \tfrac{a_{th4}\sin\left(\dot{\phi}_{th4}(t-t_3)\right)}{\dot{\phi}_{th4}}, \\ r_{net}(t) &= l_{net} + \dot{r}_{net}(t_3)(t-t_3) + \tfrac{a_{th4}\left(1-\cos\left(\dot{\phi}_{th4}(t-t_3)\right)\right)}{\dot{\phi}^2_{th4}}, \\ \dot{z}_{net}(t) &= \dot{z}_{net}(t_3) - g(t-t_3) - \tfrac{a_{th4}\left(1-\cos\left(\dot{\phi}_{th4}(t-t_3)\right)\right)}{\dot{\phi}_{th4}}, \\ z_{net}(t) &= z_{net}(t_3) + \dot{z}_{net}(t_3)(t-t_3) - \tfrac{1}{2}g(t-t_3)^2 + \tfrac{a_{th4}\left(\sin\left(\dot{\phi}_{th4}(t-t_3)\right) - \dot{\phi}_{th4}(t-t_3)\right)}{\dot{\phi}^2_{th4}}. \end{aligned} \tag{60}$$

The three final conditions to be satisfied are

$$\begin{aligned} r_{net}(t_f) &= r_f, \\ \dot{r}_{net}(t_f) &= 0, \\ \phi(t_f) &= \phi_f. \end{aligned} \tag{61}$$

By inserting (60) into (61), and solving for the three unknowns a_{th4}, $\dot{\phi}_{th4}$, and t_f, we get

$$\begin{aligned} a_{th4} &= \frac{\dot{r}_{net}(t_3)^2 \sec(\phi_{t_f})(\pi - 2\phi_{t_f} - 2\sec(\phi_{t_f}) + 2\tan(\phi_{t_f}))}{2(l_{net} - r_f)}, \\ \dot{\phi}_{th4} &= \frac{\dot{r}_{net}(t_3)(\pi - 2\phi_{t_f} - 2\sec(\phi_{t_f}) + 2\tan(\phi_{t_f}))}{2(l_{net} - r_f)}, \\ t_f &= t_3 + \frac{\frac{\pi}{2} - \phi_{t_f}}{-\dot{\phi}_{th4}}. \end{aligned} \tag{62}$$